Magnetohydroelectric waves in a fluid dielectric

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A theoretical wave behavior in a magnetized fluid dielectric is deduced. The wave consists of oscillations of magnetic, kinetic, and electrostatic energies.

Consider a fluid dielectric substance placed in a uniform magnetic field, $\mathbf{B}_0$, in the positive z direction of a rectangular Cartesian coordinate system. The dielectric constant of the fluid is $\chi\epsilon_0$, where χ is the polarizability and ϵ_0 is the permittivity of free space. Now suppose that an electric field $\mathbf{E}$ increasing with time is applied in the positive x direction. This gives rise to a polarization current $\mathbf{J}_1 = \chi\epsilon_0\dot{\mathbf{E}}$, and the consequent $\mathbf{J}_1 \times \mathbf{B}_0$ force accelerates the fluid in the negative y direction to an instantaneous velocity $\mathbf{u}$. The charges in the polarized molecules of the dielectric are now subject to an additional equivalent electric field $\mathbf{E}' = \mathbf{u} \times \mathbf{B}_0$ (owing to the force $\mathbf{u} \times \mathbf{B}_0$ per unit charge), and a little consideration will show that the effect of this field is to drive a current $\mathbf{J}_2 = \chi\epsilon_0\dot{\mathbf{E}}'$ in the direction opposite to $\mathbf{J}_1$. Thus, the net polarization current $\mathbf{J}$ measured in our coordinate system (in which $\mathbf{E}$ and $\mathbf{u}$ are measured) is

$$\mathbf{J} = \chi\epsilon_0 [\dot{\mathbf{E}} + (\partial/\partial t)(\mathbf{u} \times \mathbf{B}_0)]. \tag{1}$$

Now consider an electromagnetic disturbance in the fluid. We write $\mathbf{u} = u_y\hat{y}$, $\mathbf{E} = E_x\hat{x}$, $\mathbf{J} = J_x\hat{x}$, and $\mathbf{b} = b_y\hat{y}$, where $\mathbf{b}$ is the induced magnetic field. We assume $b \ll B_0$. Using Eq. (1), the two Maxwell's equations $\nabla \times \mathbf{E} = -\dot{\mathbf{b}}$ and $\nabla \times \mathbf{b} = \mu_0\mathbf{J} + \mu_0\epsilon_0\dot{\mathbf{E}}$, (where μ_0 is the magnetic permeability of free space) and Newton's law $\rho\dot{\mathbf{u}} = \mathbf{J} \times \mathbf{B}_0$ (ρ is the fluid mass density), we obtain the following relations

$$J_x = \chi\,\epsilon_0 (\dot{E}_x + \dot{u}_y B_0), \tag{2}$$

$$\partial b_y/\partial z = -\mu_0 J_x - \mu_0\epsilon_0\dot{E}_x, \tag{3}$$

$$\partial E_x/\partial z = -\partial b_y/\partial t, \tag{4}$$

$$\rho\dot{u}_y = -J_x B_0. \tag{5}$$

These equations may be solved simultaneously to give

$$\partial^2 b_y/\partial t^2 = V^2(\partial^2 b_y/\partial z^2), \tag{6}$$

with

$$V^2 = c^2\left(\frac{1}{\chi} + \frac{v_a^2}{c^2}\right)\left(1 + \frac{1}{\chi} + \frac{v_a^2}{c^2}\right)^{-1}, \tag{7}$$

where c is the velocity of light, and $v_a = B_0/(\mu_0\rho)^{1/2}$ is a well-known parameter called the Alfvén velocity.[1] Equation (6) indicates a wave propagating with a velocity V, in which $\mathbf{b}$, $\mathbf{u}$, and $\mathbf{E}$ oscillate, corresponding to oscillations in magnetic, kinetic, and electrostatic energies. This wave may therefore be termed a magnetohydroelectric wave. The solutions for b_y, E_x, and u_y in the wave are

$$b_y = b_0 \sin\omega(t - z/V), \tag{8}$$

$$E_x = b_0 V \sin\omega(t - z/V), \tag{9}$$

$$u_y = -\frac{Vb_0}{B_0}\left(\frac{v_a^2}{V^2} - \frac{v_a^2}{c^2}\right)\sin\omega\left(t - \frac{z}{V}\right), \tag{10}$$

where b_0 is an arbitrary constant, and ω is the wave frequency.

The ratio ξ of the kinetic energy density $\rho u_y^2/2$ to the magnetic energy density $b_y^2/2\mu_0$ in the wave is

$$\xi = (v_a/V)^2(1 - V^2/c^2)^2, \tag{11}$$

and the ratio η of the kinetic energy density to the electrostatic energy density $(1+\chi)\epsilon_0 E_x^2/2$ is

$$\eta = (v_d^2 v_a^2/V^4)(1 - V^2/c^2)^2, \tag{12}$$

where $v_d = c/(1+\chi)^{1/2}$ is the velocity of electromagnetic waves in the dielectric in absence of the magnetic field $\mathbf{B}_0$. Thus, for $V \sim v_a \sim v_d \ll c$, the kinetic, magnetic, and electrostatic energies have comparable magnitudes. Some examples of fluids with large values of χ are H_2O ($\chi \approx 80$), H_2O_2 ($\chi \approx 84$), D_2O ($\chi \approx 78$), and N_2H_4 ($\chi \approx 52$).[2]

From Eq. (8) one can derive the velocity $u_{y,fl}$ of the magnetic fieldlines to be

$$u_{y,fl} = -(Vb_0/B_0)\sin\omega(t - z/V). \tag{13}$$

Note that this velocity equals u_y when $V^2 \ll c^2$ (or $\chi \gg 1$) and $V \approx v_a$. In this case a state of "frozen flow"[1] exists where the fluid and the magnetic field lines move together at the same velocity. Whether or not this state is practically realizable in a dielectric fluid is questionable.

The resemblance of the wave derived here to Alfvén waves (magnetohydrodynamic waves) in a magnetized conducting fluid is superficial. There is no logical limit in which the present wave reduces to an Alfvén wave, or vice versa. More generally, we may consider a partially conducting and partially dielectric fluid, and replace Eq. (2) by

$$J_x = \chi\epsilon_0(\dot{E}_x + \dot{u}_y B_0) + \sigma(E_x + u_y B_0), \tag{14}$$

where σ is the conductivity. Equations (14) and (3)–(5) may now be solved simultaneously to yield Eq. (6) with V replaced by V' where

$$V'^2 = c^2\,\frac{\dfrac{1}{\chi} + \dfrac{v_a^2}{c^2} - i\,\dfrac{\mu_0\sigma}{\chi\omega}\,v_a^2}{1 + \dfrac{1}{\chi} + \dfrac{v_a^2}{c^2} - i\,\dfrac{\mu_0\sigma}{\chi\omega}\left(1 + \dfrac{v_a^2}{c^2}\right)c^2}, \tag{15}$$

where ω is the wave frequency. In the limit $\sigma \to 0$, we have $V' = V$, and in the limit $\chi \to 0$, $\mu_0\sigma c^2/\omega \to \infty$, we have $V' \to (v_a^{-2} + c^{-2})^{-1/2}$ which, when $v_a \ll c$, reduces to the Alfvén velocity.

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